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The generation of sound by density inhomogeneities in low Mach number nozzle flows

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This paper discusses the sound generated when an inhomogeneity in density is convected in a low Mach number steady flow through a contraction in a duct of infinite extent, and also when the inhomogeneity exhausts through a nozzle into free space. The analyses of Candel (1972) and Marble (1973) for the case of duct flow were based on a frequency decomposition of the incident inhomogeneity and cannot adequately deal with sharp-fronted inhomogeneities and entropy spots. However, the practical difficulties of this earlier work can be avoided at low flow Mach numbers by conducting the analysis in terms of an approximate expression for the acoustic Green's function in the manner described by Howe (1975). This method also permits a considerable extension of the range of the earlier investigations to the determination of the sound generated when the inhomogeneity is swept out of a nozzle orifice into free space. It is shown that the acoustic pressure perturbations developed in a duct at a contraction are in general proportional to the fractional difference between the density of the inhomogeneity and that of the mean flow times a typical mean flow pressure level, and are due principally to the fluctuation in thrust accompanying the passage of the inhomogeneity through the region of variable pressure gradient. The pressure waves generated at a nozzle orifice and radiated into free space are $O(M_0)$ smaller, where M_0 is a mean flow Mach number based on the speed of sound in the jet.

1. Introduction

The recent work of Candel (1972), Marble (1973) and Cumpsty & Marble (1974) has demonstrated that sound is generated when a nominally *silent* fluid inhomogeneity, such as an entropy spot, is convected through a region of non-uniform flow. This is because the inhomogeneity cannot negotiate a path through the existing non-uniform mean flow without the intervention of a compensating pressure perturbation. This perturbation manifests itself in the form of an acoustic pulse.

The noise produced by this mechanism has been analysed by first decomposing the incident inhomogeneity into a Fourier distribution of entropy waves. The interaction of each component with the non-uniformity in the flow, e.g. with a sudden or gradual change in duct cross-section, is then worked out in detail and the results can, in principle, be superposed to yield the effect of a specific inhomogeneity. When the contraction in the duct is treated as a compact element (Marble 1973), such that its length and transit time are small compared with the wavelength and wave period, this theory is only capable of dealing with diffuse entropy inhomogeneities in which temperature and density variations occur over distances greater than the scale of the contraction region. The problem of an incident entropy *spot* has been considered by Candel (1972) and Marble (1973) by means of a *one-dimensional* model in which the mean flow characteristics are functions of a single axial co-ordinate. Actually both authors assumed that the variation in mean flow velocity is a linear function of position. However, their analysis is again based on a frequency decomposition of the incident inhomogeneity and the ensuing mathematical complications are such as to inhibit a detailed investigation other than by way of numerical computation. The difficulty arises because the method of Fourier decomposing the incident entropy spot is particularly inefficient for this type of problem. The dominant frequency of the emitted sound is determined essentially by the time taken by the spot to convect through the region of non-uniform flow, and the summation of the Fourier components constituting this sound must be undertaken with more precision than is generally possible by numerical computation in order that the delicate phase cancellation of the high frequency components present in the incident entropy spot can be achieved. Thus it is not possible by this means to obtain analytical expressions which enable one to chart the progress of an individual entropy inhomogeneity through a nozzle contraction, to elicit in detail the physical mechanisms responsible for the generation of sound, nor to predict the wave form of the emitted sound.

An approach proposed by Ffowcs Williams (1974), on the other hand, is designed specifically to expose the noise source mechanisms that are possible in a region of non-uniform flow. Indeed he showed that the free-space far-field acoustic pressure perturbation p induced by a compact source region V located near the origin $\mathbf{x} = 0$ can be expressed in the form

$$p(\mathbf{x},t) = \frac{1}{4\pi c_0^2}\frac{x_i x_j}{|\mathbf{x}|^3}\int_V \left[\rho \frac{D}{Dt}\left\{\frac{1}{(1-M_r)}\frac{D}{Dt}\left(\frac{T_{ij}}{\rho(1-M_r)}\right)\right\}\right] d^3\mathbf{y}, \tag{1.1}$$

where ρ, c_0 and T_{ij} are respectively the fluid density, the speed of sound in free space and the Lighthill stress tensor (Ffowcs Williams 1974, equation (3)) and the integral is evaluated at the retarded time $t-|\mathbf{x}-\mathbf{y}|/c_0$. The Mach number M_r is equal to U_r/c_0, where U_r is the speed at which the fluid element approaches the observation point $\mathbf{x}$.

The material derivatives D/Dt in the result (1.1) serve to emphasize that it is time variations relative to a Lagrangian reference frame convecting with the fluid which actually determine the characteristics of the radiation. A nominally 'silent' source is one for which the Lighthill tensor T_{ij} is constant or changes very slowly in the source-fixed frame. Equation (1.1) then predicts that the radiated pressure levels are negligible unless the source, which is attached to a fluid particle, is itself accelerating, e.g. through a region of non-uniform mean flow, in which case the dominant radiation may be termed *acoustic bremsstrahlung*, and for low convection Mach numbers is given by

$$p(\mathbf{x},t) \simeq \frac{1}{4\pi c_0^2}\frac{x_i x_j}{|\mathbf{x}|^3}\int_V \left[T_{ij}\frac{D^2}{Dt^2}M_r\right] d^3\mathbf{y}. \tag{1.2}$$

In particular, the case of an entropy spot is included in a model where

$$T_{ij} \simeq c_0^2(\rho_0 - \rho)\,\delta_{ij}, \tag{1.3}$$

ρ being the density of a particular fluid particle; (1.2) then gives

$$p(\mathbf{x},t) = \frac{1}{4\pi c_0}\frac{x_j}{|\mathbf{x}|^2}\int_V \left[\frac{D}{Dt}\left\{\frac{(\rho-\rho_0)}{\rho}\frac{\partial p}{\partial y_j}\right\}\right] d^3\mathbf{y}, \tag{1.4}$$

the momentum equation $\rho Du_i/Dt = -\partial p/\partial x_i$ having been invoked to simplify further its integrand.

In this form the bremsstrahlung is seen to be proportional to $(\rho-\rho_0)\,M_0 U^2$, where U is a typical convection speed and $M_0 = U/c_0$. The noise source mechanism is the interaction between the density inhomogeneity $\rho-\rho_0$ and the mean flow pressure gradient, a source term which has already been identified by Morfey (1973) by an alternative procedure.

The general free-space result (1.1) obtained by Ffowcs Williams is not directly applicable to duct and nozzle flow problems because of the added complexity introduced by the presence of dipole source terms distributed over the walls of the duct and the nozzle lip. However, in the case of low Mach number flows, it is possible to avoid the analytical difficulties experienced by Candel (1972) and Marble (1973) in their treatment of the variable duct, by first deriving an approximate expression for the Green's function describing the generation of sound within the variable-geometry duct or nozzle. A procedure for obtaining such Green's functions has been described by Howe (1975), and this enables one to construct an elegant analytical theory for the Candel–Marble problem of the sound generated during the convection through a contraction in a duct of an entropy inhomogeneity, and also to extend the range of the investigation to determine the sound generated when the inhomogeneity is swept out of a nozzle orifice into free space. This theory is described below.

In §2 attention is focused on the problem of determining the sound generated when an inhomogeneity is convected through a contraction in an infinitely long duct, i.e. on the case of flow interaction far upstream of a nozzle orifice. It is convenient to consider separately two extreme possibilities. The first is that of a *slug* of material convected at low Mach number through the contraction. A pressure perturbation is developed across the slug by precisely the same mechanism as that described above in (1.4), the radiated pressure level being proportional to the total mean pressure drop across the slug and to the fractional density difference between the slug and the ambient fluid. This is the dominant acoustic source mechanism. The contribution to the sound field produced as a result of the differential change in the volume of the slug as it convects down the mean flow pressure gradient is *smaller* than the bremsstrahlung term by a factor of the order of the Mach number of the mean flow.

A small spherical *pellet* which is convected along the duct behaves as a *dipole* source of acoustic radiation during the period in which it is slipping through the fluid. This occurs where the flow is non-uniform, and the perturbation pressure levels developed are proportional to the pellet velocity and the slip velocity together with factors characterizing the geometry of the mean flow.

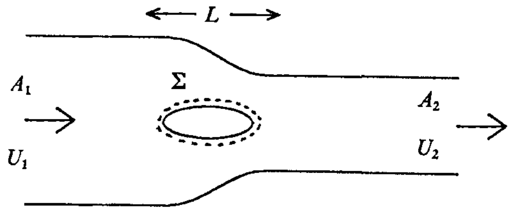

FIGURE 1. Hard-walled duct. The flow inhomogeneity is bounded by a control surface Σ which lies just within the region of validity of the wave equation (2.1).

In both of these model problems the acoustic pressure fluctuations are typically of order $\rho_0 U^2$. At a nozzle orifice, however, the sound radiated into free space scales on the dimensionless source frequency times ρU^2, and that frequency is of order M_0, the characteristic flow Mach number (§3), so that the free-space radiation is $O(\rho M_0 U^2)$. Further calculation (§4) of the respective cases of a slug and a pellet emerging from a nozzle orifice into free space confirms this general prediction regarding the overall free-space sound pressure level.

2. Convection through a duct contraction

Consider the situation depicted in figure 1. A hard-walled duct of infinite length contains fluid of density ρ_0 in a state of steady flow. The flow, which is in the $+x_1$ direction, accelerates through a contraction of scale L over which the uniform cross-sectional area of the duct reduces from A_1 to A_2 and the mean flow velocity increases from U_1 to U_2. The mean flow Mach number is assumed to be sufficiently small that the steady flow may be regarded as incompressible, with $A_1 U_1 = A_2 U_2$.

An inhomogeneity present in the flow will be accelerated through the contraction in a time of order L/U, resulting in the emission of sound waves of wavelength $O(L/M)$, M being the characteristic flow Mach number. For sufficiently small values of M the wavelength will greatly exceed the contraction scale L, and the problem of determining the radiated sound pressure levels can then be analysed by means of the low Mach number theory developed in Howe (1975).

We assume that the steady flow is irrotational. Let ϕ denote the perturbation velocity potential produced by the inhomogeneity. Then for points exterior to the inhomogeneity ϕ satisfies the convected wave equation

$$\frac{1}{c_0^2}\left(\frac{\partial}{\partial t}+\mathbf{U}.\frac{\partial}{\partial \mathbf{x}}\right)^2 \phi-\nabla^2\phi = 0, \tag{2.1}$$

provided that $M^2 \ll 1$. In this result the steady flow velocity $\mathbf{U}$ is variable only over the region of the contraction, and $\mathbf{U} \to (U_1, 0, 0), (U_2, 0, 0)$ as $x_1 \to \mp \infty$.

Next enclose the flow inhomogeneity within a closed surface Σ lying just within the region of validity of (2.1), and defined by $f(\mathbf{x}, t) = 0$ (see figure 1). Assume that $f > 0$ for points of the flow outside Σ and $f < 0$ within Σ, and in the manner of Ffowcs Williams & Hawkings (1969), set $\psi = H(f)\phi$, where H

is the Heaviside unit function. Multiply (2.1) by $H(f)$ and rearrange to obtain

$$\frac{1}{c_0^2}\left(\frac{\partial}{\partial t}+\mathbf{U}\cdot\frac{\partial}{\partial \mathbf{x}}\right)^2\psi-\nabla^2\psi=\frac{1}{c_0^2}\left(\frac{\partial}{\partial t}+\mathbf{U}\cdot\frac{\partial}{\partial \mathbf{x}}\right)\left\{\phi\left(\frac{\partial}{\partial t}+\mathbf{U}\cdot\frac{\partial}{\partial \mathbf{x}}\right)H\right\}$$
$$+\frac{1}{c_0^2}\left(\frac{\partial}{\partial t}+\mathbf{U}\cdot\frac{\partial}{\partial \mathbf{x}}\right)\phi\cdot\left(\frac{\partial}{\partial t}+\mathbf{U}\cdot\frac{\partial}{\partial \mathbf{x}}\right)H-\frac{\partial}{\partial x_j}\left(\phi\frac{\partial H}{\partial x_j}\right)-\frac{\partial \phi}{\partial x_j}\frac{\partial H}{\partial x_j}. \quad (2.2)$$

This is an equation for ψ which is formally valid throughout the whole of the duct, with $\psi=\phi$ for points exterior to the surface Σ. If the *source* terms on the right-hand side are known, (2.2) can be solved for ψ as soon as the Green's function for the duct is specified. This is the solution of the equation

$$\frac{1}{c_0^2}\left(\frac{\partial}{\partial t}+\mathbf{U}\cdot\frac{\partial}{\partial \mathbf{x}}\right)^2 G-\nabla^2 G=\delta(t-\tau)\,\delta(\mathbf{x}-\mathbf{y}), \quad (2.3)$$

whose normal derivative vanishes on the walls of the duct and which satisfies the radiation condition at infinity.

The form of $G(\mathbf{x},\mathbf{y};t,\tau)$ appropriate to the low frequency source distributions under consideration can be obtained by means of a straightforward application of the method described by Howe (1975), and outlined in the appendix. Actually we shall confine our attention to the determination of the sound radiated downstream of the contraction ($x_1>0$). In this case we have

$$G(\mathbf{x},\mathbf{y};t,\tau)=\frac{c_0}{A_1+A_2}H\left\{t-\tau-\frac{x_1}{c_0(1+M_2)}+\frac{A_1}{A_2}\frac{\phi^*(\mathbf{y})}{c_0(1+M_1)}\right\}, \quad (2.4)$$

where $M_1=U_1/c_0$, $M_2=U_2/c_0$ and $\phi^*(\mathbf{x})$ is a harmonic function describing irrotational flow through the contraction, and normalized such that

$$\phi^*\to x_1 \quad \text{as} \quad x_1\to+\infty.$$

It is assumed in (2.4) that $\mathbf{y}$ is located in the region of the contraction and that the observation point $\mathbf{x}$ is far downstream.

Let us first consider the case of the downstream radiation produced as a result of the passage of a slug of material of density ρ through the contraction, where $(\rho-\rho_0)/\rho_0$ is small (see figure 2). For sufficiently small flow Mach numbers the acoustic wavelength will greatly exceed the dimensions of the slug, and the flow in the vicinity of the contraction will be essentially incompressible.

Actually for this particular problem it is convenient to work in terms of $\partial\phi/\partial t$ rather than ϕ, in which case the equation analogous to (2.2) is

$$\frac{1}{c_0^2}\left(\frac{\partial}{\partial t}+\mathbf{U}\cdot\frac{\partial}{\partial \mathbf{x}}\right)^2\psi-\nabla^2\psi=-\frac{\partial}{\partial x_j}\left(\dot\phi\frac{\partial H}{\partial x_j}\right)-\frac{\partial \dot\phi}{\partial x_j}\frac{\partial H}{\partial x_j}. \quad (2.5)$$

Here we have discarded the material-derivative source terms which appear on the right-hand side of (2.2), since under the present circumstances they are $O(M^2)$ smaller than the terms retained on the right of (2.5).

Since the flow regime is effectively incompressible in the region of the contraction, we can set

$$\phi=m(t)+[U_2+u(t)]\,\phi^*(\mathbf{x})+\dots, \quad (2.6)$$

where $U_2+u(t)$ is the uniform flow velocity downstream at distances large compared with the contraction scale L but small compared with the acoustic

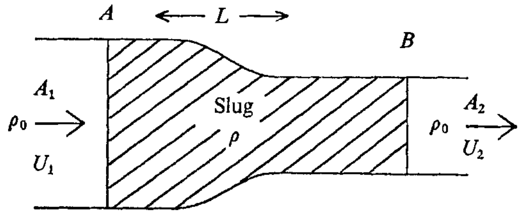

FIGURE 2. A slug of material of density ρ is convected by the mean flow through the contraction in the duct.

wavelength. The purely time-dependent contribution $m(t)$ is discontinuous across the end faces of the slug, and this is the dominant source of the acoustic radiation.

The downstream radiation from the slug can be estimated by substituting (2.6) into the source terms on the right-hand side of (2.5) and then convoluting with the Green's function (2.4). Consider first the contribution from the dipole source term, viz.,

$$\psi_d = \frac{-c_0}{A_1+A_2}\int \frac{\partial}{\partial y_j}\left(\dot{\phi}\frac{\partial H}{\partial y_j}\right) H\left\{t-\tau-\frac{x_1}{c_0(1+M_2)}+\frac{A_1}{A_2}\frac{\phi^*(\mathbf{y})}{c_0(1+M_1)}\right\} d^3\mathbf{y}\,d\tau$$
$$\simeq \frac{A_1}{A_2(A_1+A_2)(1+M_1)}\int \dot{\phi}(\mathbf{y},\tau)\frac{\partial H}{\partial y_j}\frac{\partial \phi^*}{\partial y_j}\delta\left\{t-\tau-\frac{x_1}{c_0(1+M_2)}\right\} d^3\mathbf{y}\,d\tau, \quad (2.7)$$

where the effective differences in retarded times over the length of the slug have been neglected. Neglecting also the variations in $\dot{\phi}(\mathbf{y},\tau)$ over the end faces of the slug we have

$$\psi_d \simeq \frac{A_1}{(A_1+A_2)(1+M_1)}\left[\dot{\phi}\left(\mathbf{y}, t-\frac{x_1}{c_0(1+M_2)}\right)\right]_A^B, \quad (2.8)$$

where $[\dot{\phi}]_A^B$ is to be evaluated at points just *outside* the slug at the stations A and B shown in figure 2.

Consider next the monopole term

$$\psi_m = \frac{-c_0}{A_1+A_2}\int \frac{\partial \dot{\phi}}{\partial y_j}\frac{\partial H}{\partial y_j} H\left\{t-\tau-\frac{x_1}{c_0(1+M_2)}+\frac{A_1}{A_2}\frac{\phi^*(\mathbf{y})}{c_0(1+M_1)}\right\} d^3\mathbf{y}\,d\tau. \quad (2.9)$$

Using (2.6) and observing that $\nabla^2\phi^* = 0$ we have

$$\psi_m \simeq \frac{-c_0}{A_1+A_2}\int \dot{u}(\tau)\frac{\partial \phi^*}{\partial y_j}\frac{\partial H}{\partial y_j}\left\{H\left(t-\tau-\frac{x_1}{c_0(1+M_2)}\right) + \frac{A_1}{A_2}\frac{\phi^*(\mathbf{y})}{c_0(1+M_1)}\delta\left(t-\tau-\frac{x_1}{c_0(1+M_2)}\right)\right\} d^3\mathbf{y}\,d\tau$$
$$\simeq \frac{-A_1}{(A_1+A_2)(1+M_1)}\dot{u}\left(t-\frac{x_1}{c(1+M_2)}\right)[\phi^*(\mathbf{y})]_A^B. \quad (2.10)$$

Hence

$$\psi = \psi_d+\psi_m \simeq \frac{A_1}{(A_1+A_2)(1+M_1)}\left[\dot{m}\left(t-\frac{x_1}{c_0(1+M_2)}\right)\right]_A^B. \quad (2.11)$$

Now in the absence of the viscous generation of vorticity, Bernoulli's equation assumes the form

$$\nabla\{\phi+\tfrac{1}{2}V^2+\textstyle\int dp/\rho\} = 0 \tag{2.12}$$

in each region of uniform mean density, from which, availing ourselves of the continuity of pressure and of velocity (assumed to be normal to the end faces of the slug), we find

$$\left[\dot{m}\left(t-\frac{x_1}{c_0(1+M_2)}\right)\right]_A^B = \frac{(\rho-\rho_0)}{\rho\rho_0}(p_A-p_B). \tag{2.13}$$

In the first approximation p_A and p_B may be replaced by their values in the absence of the slug, so that, noting further that the perturbation pressure p downstream of the contraction is related to ψ by

$$p \simeq -\rho_0(1+M_2)^{-1}\dot{\psi}, \tag{2.14}$$

we finally deduce that

$$p \simeq \frac{-1}{(1+M_1)(1+M_2)}\left(\frac{A_1}{A_1+A_2}\right)\left(\frac{\rho-\rho_0}{\rho}\right)[p_A-p_B]_T, \tag{2.15}$$

where $T = t-x_1/c_0(1+M_2)$.

This result characterizes the respective effects of the flow, area change, density ratio and of the mean flow pressure drop across the slug. Since

$$p_A-p_B \simeq l\,\partial p/\partial x,$$

where l is of the order of the length of the slug, it is apparent that the source of the acoustic radiation can be identified with the bremsstrahlung dipole source term arising from the interaction of the density inhomogeneity and the mean flow pressure gradient in the corresponding free-space result (1.4). This conclusion is borne out by detailed calculation along the above lines in which the free-space Green's function is used.

Actually acoustic emission may also occur as a result of the volumetric change as the slug convects into a region of lower ambient pressure. The acoustic response is a consequence of this change in volume being different from the change in the volume of a corresponding hypothetical slug made up of ambient fluid matter. In order to estimate the magnitude of the sound generated and to compare it with the dipole term isolated in (2.15), we first note that, if $\Box^2$ denotes the convective wave operator on the left of (2.1), then the difference between the response of the material of the slug and that of the ambient fluid arises because $\Box^2\phi \neq 0$ within the slug. Indeed, if c is the sound speed within the slug, and $\Box_s^2$ the corresponding wave operator, then for points within the material of the slug

$$\Box_s^2\phi = 0. \tag{2.16}$$

By means of the Heaviside function $H(f)$ introduced above we can formally write

$$\Box^2\phi = (1-H)\,\Box^2\phi \equiv \partial\{(1-H)\,\Box^2\phi\}/\partial t, \tag{2.17}$$

since $\Box^2\phi = 0$ on $f(\mathbf{x},t) = 0$, i.e. on the control surface Σ. The right-hand side of this equation is non-zero within Σ for two reasons. First, ϕ is discontinuous across the boundaries of the slug, giving rise to surface distributions of acoustic source terms which have already been examined above. Second, within the volume of the slug $\Box^2\phi \neq 0$, but $\Box_s\phi^2 = 0$.

Thus (2.17) may be set in the more explicit form

$$\Box^2\phi = -\frac{\partial}{\partial t}\left\{\frac{\partial}{\partial x_j}[\phi]\frac{\partial H}{\partial x_j}\right\}+\frac{\partial}{\partial t}\left\{(1-H)\left(\frac{c^2-c_0^2}{c_0^2}\right)\nabla^2\phi\right\}, \tag{2.18}$$

where $f(\mathbf{x}, t) = 0$ now defines the boundary of the slug and $[\phi]$ denotes the jump in the value of ϕ in passing out of the slug. Noting that within the slug $\nabla^2\phi = -(\rho c^2)^{-1} Dp/Dt$, the contribution, p_m say, to the downstream radiation from the second term on the right of (2.18) is obtained by convolution with the Green's function (2.4):

$$p_m \simeq \frac{\Delta}{(A_1+A_2)}\frac{\rho_0}{\rho}\left(1-\frac{c_0^2}{c^2}\right)[\mathbf{M}_0.\nabla p]_T, \tag{2.19}$$

where Δ is the volume of the slug and $T = t - x_1/c_0(1+M_2)$.

The surface term in (2.18) produces the dipole radiation field given by (2.15). But when account is taken of the small difference in the retarded times of p_A and p_B there is an additional contribution p_D to (2.15) which is of the same order as p_m, viz.

$$p_D \simeq \frac{\Delta}{(A_1+A_2)}\left(1-\frac{\rho_0}{\rho}\right)[\mathbf{M}_0.\nabla p]_T. \tag{2.20}$$

Combining (2.19) and (2.20) it follows that the downstream radiation arising from the differential volumetric response of the slug is given by

$$p \simeq \frac{\Delta}{(A_1+A_2)}\left(\frac{\rho c^2-\rho_0 c_0^2}{\rho c^2}\right)[\mathbf{M}_0.\nabla p]_T. \tag{2.21}$$

Let us now consider two extreme applications of this result. The first is to the case of an *entropy* inhomogeneity in which the slug actually consists of a portion of fluid within which the temperature differs from that of the steady mean flow. The ratio of the specific heats may then be assumed to be constant throughout the whole of the fluid, so that $\rho c^2 = \rho_0 c_0^2$. This is therefore a case in which the compressibility of the slug is the same as that of the ambient flow, and *no* sound is generated by this mechanism.

The second application of the volumetric source term of (2.18) is to the case in which the change in the volume of the slug as it convects through the contraction in the duct is guaranteed to be large. This occurs, for example, when the slug consists of a 'spongy' mass of gas convected in liquid, typified by an air bubble in a domestic hose. In that case, the compressibility of the air being much greater than that of water, (2.21) implies a downstream pressure perturbation whose order of magnitude is given by

$$p \sim \frac{-\Delta}{(1+M_2)(A_1+A_2)}\left(\frac{\rho_0 c_0^2}{\rho c^2}\right)\left[M_0\frac{\partial p}{\partial x_1}\right], \tag{2.22}$$

where, as before $M_0 = U/c_0$. This result is $O(M_0)$ relative to the dipole interaction term (2.15), but contains the large factor $\rho_0 c_0^2/\rho c^2$, which in the case of an air bubble in water is of order 10^5. Comparison with the dipole term indicates that (2.22) is significant provided that

$$M_0 = U/c_0 \gtrsim (c/c_0)^2.$$

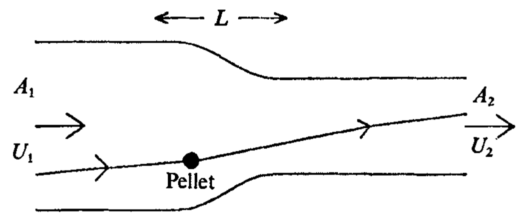

FIGURE 3. A small spherical pellet emits sound as it accelerates through the contraction in the duct.

The ratio on the right-hand side is about 0·05 for an air bubble in water, and the condition then corresponds to flow speeds greater than about 75 m s^{-1}, which is much larger than those occurring in a domestic hose.

We now turn our attention to the problem of determining the pressure waves developed during the convection of a spherical *pellet* through the contraction (figure 3). At low relative flow Mach numbers the disturbed flow in the region of the pellet is incompressible. Thus, if the pellet is sufficiently small and has velocity of translation $\mathbf{V}$ and, as before, $\mathbf{U}(\mathbf{x})$ denotes the steady ambient velocity field, then the perturbation *potential* in the vicinity of the pellet is given by

$$\phi_p = -\frac{a^3(\mathbf{x}-\mathbf{x}_0(t))}{2r^3}\cdot(\mathbf{V}-\mathbf{U}). \tag{2.23}$$

In this equation a is the radius of the pellet, $\mathbf{x}_0(t)$ the location of its centre at time t, and $r = |\mathbf{x}-\mathbf{x}_0(t)|$. The wave equation analogous to (2.5) is then

$$\frac{1}{c_0^2}\left(\frac{\partial}{\partial t}+\mathbf{U}\cdot\frac{\partial}{\partial \mathbf{x}}\right)^2\psi-\nabla^2\psi = -\frac{\partial}{\partial x_j}\left(\phi_p\frac{\partial H}{\partial x_j}\right)-\frac{\partial \phi_p}{\partial x_j}\frac{\partial H}{\partial x_j}, \tag{2.24}$$

where the control surface Σ may be assumed to coincide with the surface of the pellet.

It is convenient to adopt the following procedure. Rather than solving (2.24) by direct convolution of the right-hand side with the Green's function (2.4), we first simplify the source terms by means of a formal expansion in powers of the radius a of the pellet. This is equivalent to an expansion of the actual solution in powers of a compactness parameter Ma/L, where M is the characteristic flow Mach number and L is the length scale of the contraction. To do this we introduce a 'test' function $f(\mathbf{x})$, say. Then the volume integral I_D of the product of the dipole source term and the test function is

$$I_D = -\int\frac{\partial}{\partial x_j}\left(\phi_p\frac{\partial H}{\partial x_j}\right)f(\mathbf{x})\,d^3\mathbf{x} = \int\phi_p\frac{\partial f}{\partial x_j}\frac{\partial H}{\partial x_j}\,d^3\mathbf{x}. \tag{2.25}$$

Expand $\partial f/\partial x_j$ in powers of $x_i - x_{0i}$, i.e. about the centre of the sphere, and retain only the first non-trivial term. Using the expression (2.23) we readily deduce that in the first approximation

$$I_D \simeq -\tfrac{2}{3}\pi a^3(\mathbf{V}-\mathbf{U})\cdot\partial f/\partial\mathbf{x}, \tag{2.26}$$

from which it follows that the dipole source 'strength' is represented by

$$-\frac{\partial}{\partial x_j}\left(\phi_p\frac{\partial H}{\partial x_j}\right) \simeq \frac{2\pi a^3}{3}\operatorname{div}\{(\mathbf{V}-\mathbf{U})\,\delta[\mathbf{x}-\mathbf{x}_0(t)]\}. \tag{2.27}$$

A similar calculation applied to the monopole term leads to

$$-\frac{\partial \phi_p}{\partial x_j}\frac{\partial H}{\partial x_j} \simeq \frac{4\pi a^3}{3}\operatorname{div}\{(\mathbf{V}-\mathbf{U})\,\delta[\mathbf{x}-\mathbf{x}_0(t)]\}. \tag{2.28}$$

Hence the principal net effect of the pellet is that of a convected dipole source, and the corresponding acoustic perturbation potential ϕ is determined by

$$\frac{1}{c_0^2}\left(\frac{\partial}{\partial t}+\mathbf{U}.\frac{\partial}{\partial \mathbf{x}}\right)^2 \phi - \nabla^2\phi = 2\pi a^3 \operatorname{div}\{(\mathbf{V}-\mathbf{U})\,\delta[\mathbf{x}-\mathbf{x}_0(t)]\}. \tag{2.29}$$

Convolution of the right-hand side of this equation with the Green's function (2.4) is now trivial, and yields for the downstream perturbation potential

$$\begin{aligned}\phi &\simeq \frac{-2\pi a^3 A_1}{A_2(A_1+A_2)(1+M_1)}\int (\mathbf{V}-\mathbf{U}).\frac{\partial \phi^*}{\partial \mathbf{y}}\,\delta[\mathbf{y}-\mathbf{x}_0(\tau)]\,\delta\left\{t-\tau-\frac{x_1}{c_0(1+M_2)}\right\} d^3\mathbf{y}\,d\tau \\ &= \frac{-2\pi a^3 A_1}{A_2(A_1+A_2)(1+M_1)}[(\mathbf{V}-\mathbf{U}).\nabla\phi^*]\end{aligned} \tag{2.30}$$

evaluated at $\mathbf{x} = \mathbf{x}_0(t-x_1/c_0(1+M_2))$.

The acoustic pressure perturbation is related to ϕ by

$$p \simeq -\frac{\rho_0}{1+M_2}\frac{\partial \phi}{\partial t} \simeq \frac{-\rho_0}{1+M_2}\left[\mathbf{V}.\frac{\partial \phi}{\partial \mathbf{x}}\right]_{\mathbf{x}=\mathbf{x}_0}, \tag{2.31}$$

from which it follows that

$$p \simeq \frac{2\pi a^3 A_1 \rho_0}{A_2(A_1+A_2)(1+M_1)(1+M_2)}\left[\mathbf{V}.\frac{\partial}{\partial \mathbf{x}}\left\{(\mathbf{V}-\mathbf{U}).\frac{\partial \phi^*}{\partial \mathbf{x}}\right\}\right] \tag{2.32}$$

evaluated at $\mathbf{x} = \mathbf{x}_0(t-x_1/c_0(1+M_2))$.

Again, therefore, the radiation pressure field is proportional to the square of a typical flow velocity, sound being emitted only during the period in which the pellet is slipping through the ambient flow field.

3. Free-space radiation

In §2 the acoustic field generated by the passage of a density inhomogeneity through a contraction in a duct was calculated on the assumption that the duct was of infinite extent in both directions. We now relate this to the radiation emitted into free space at a nozzle termination, assumed to be located far downstream of the contraction at which the sound originates.

The simple order-of-magnitude calculation, which is all that is really required here, is classical and is given in various forms in Rayleigh (1945, chap. 16). Here we neglect the effect of the low Mach number flow, and consider a long plane wave of potential $\phi_I(t-x_1/c_0)$ incident on the nozzle exit from the interior (see figure 4). The potential of the field radiated into free space is given approximately by

$$\phi \simeq 2\left(\frac{l}{r}\right)\frac{l}{c_0}\frac{\partial}{\partial t}\phi_I\left(t-\frac{r}{c}\right), \tag{3.1}$$

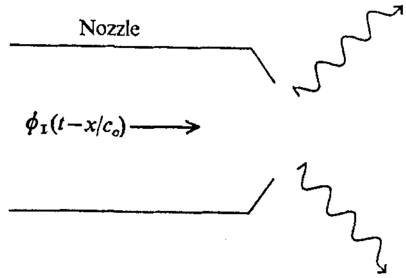

FIGURE 4. Scattering of a long wave incident on the nozzle from the interior of the duct.

where r is the observation distance from the orifice, l is determined by the exact details of the nozzle construction, but is a length whose magnitude is of the order of half the orifice radius, and c_s is the free-space sound speed.

The far-field pressure perturbation is

$$p \simeq -2\rho_s\left(\frac{l}{r}\right)\frac{l}{c_0}\frac{\partial^2}{\partial t^2}\phi_I\left(t-\frac{r}{c_s}\right). \tag{3.2}$$

Now the considerations of §2 indicate that in both of the extreme cases of a slug and of a pellet convecting in the flow

$$\partial\phi_I/\partial t = O(U^2), \tag{3.3}$$

and since the time scale of the acoustic disturbance is $O(L/U)$, this implies that the free-space radiated pressure field is typically of order

$$p = -2\rho_s\left(\frac{l}{L}\right)\left(\frac{l}{r}\right)O(M_0 U^2). \tag{3.4}$$

Note that the Mach number dependence of this result is with respect to the speed of sound *within* the duct flow.

In particular if $\mathscr{A}$ is the cross-sectional area of the duct upstream of the nozzle but, as in figure 1, downstream of the duct contraction considered in §2, and if it is assumed that $l^2 = \mathscr{A}/4\pi$, then the dipole radiation (2.15) within the duct gives rise to free-space perturbation pressures of the order

$$\begin{aligned} p &\sim \frac{-A_1}{2\pi(A_1+A_2)}\left(\frac{\rho_s}{\rho_0}\right)\left(\frac{\rho-\rho_0}{\rho}\right)\frac{\mathscr{A}}{rc_0}\frac{\partial}{\partial t}[p_A-p_B] \\ &\sim \frac{-A_1}{2\pi(A_1+A_2)}\left(\frac{\rho_s}{\rho_0}\right)\left(\frac{\rho-\rho_0}{\rho}\right)\frac{\mathscr{A}}{rL}\frac{U}{c_0}[p_A-p_B]. \end{aligned} \tag{3.5}$$

This result will be considered further in the next section.

4. Generation of sound at the nozzle orifice

The slug and pellet calculations of §2 can be repeated for the case of a *terminating* duct, although it is difficult to take direct account of the vortex shear layer which separates the nozzle jet flow from the ambient atmosphere. However, in the long-wavelength approximation under consideration, it might be

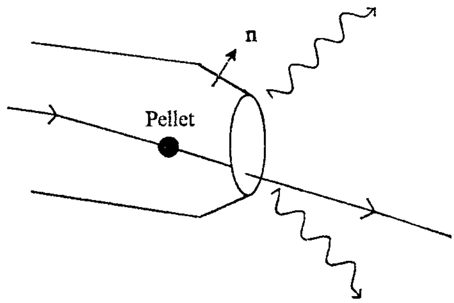

FIGURE 5. Acoustic radiation is emitted as a pellet is swept out of the nozzle by the jet flow.

anticipated that the neglect of shear-layer effects will not lead to substantial deviations of the solution of such a simplified problem from that of the actual physical problem which it is desired to model.

Consider the typical axisymmetric nozzle configuration illustrated in figure 5, which has uniform cross-sectional area $\mathscr{A}$ far upstream of the orifice. For sufficiently low exhaust Mach numbers the Green's function for the nozzle can be determined in the manner already outlined in the appendix in connexion with the infinite duct. This is just the low frequency approximation to the exact Green's function, and for an observation point $\mathbf{x}$ in free space and many wavelengths from the orifice, it can be set in the form

$$G(\mathbf{x},\mathbf{y};t,\tau) = \frac{1}{4\pi|\mathbf{x}|}\,\delta\left\{t-\tau-\frac{[|\mathbf{x}-\mathbf{K}(\mathbf{y})|-F(\mathbf{y})]}{c_s}\right\}, \tag{4.1}$$

where the origin of the co-ordinates is located in the centre of the orifice and c_s is the speed of sound in the exterior fluid.

The functions $F(\mathbf{y})$ and $\mathbf{K}(\mathbf{y})$ are harmonic and have the dimensions of length. They characterize respectively the monopole and dipole responses of the nozzle, which are of equal importance at low frequencies, and satisfy the normal-velocity condition

$$\mathbf{n}.\nabla(F,\mathbf{K}) = 0 \tag{4.2}$$

on the nozzle wall, $\mathbf{n}$ being the unit normal illustrated in figure 5. As $|\mathbf{y}|\to\infty$ in free space, $\mathbf{K}\to\mathbf{y}$, $F\to 0$ and (4.1) reduces to the usual free-space Green's function in which *Rayleigh* scattering by the nozzle is neglected. As $|\mathbf{y}|\to\infty$ within the duct, $\mathbf{K}\to$ constant, $F\to(c_s/c_0)\,y_1$, where c_0 is the speed of sound in the jet flow, and we recover the Green's function for a low frequency source located far upstream of the orifice. Further details of the functions F and $\mathbf{K}$ for the particular case of a circular cylindrical nozzle are given by Leppington (1971).

Consider first the case of a spherical pellet exhausting through the nozzle. Using the dipole source term (2.29) we obtain the perturbation potential in free space:

$$\phi \simeq \frac{a^3}{2|\mathbf{x}|}\int\frac{\partial}{\partial y_j}\{(V_j-U_j)\,\delta(\mathbf{y}-\mathbf{x}_0(\tau))\}\,\delta\left\{t-\tau-\frac{[|\mathbf{x}-\mathbf{K}(\mathbf{y})|-F(\mathbf{y})]}{c_s}\right\}d^3\mathbf{y}\,d\tau, \tag{4.3}$$

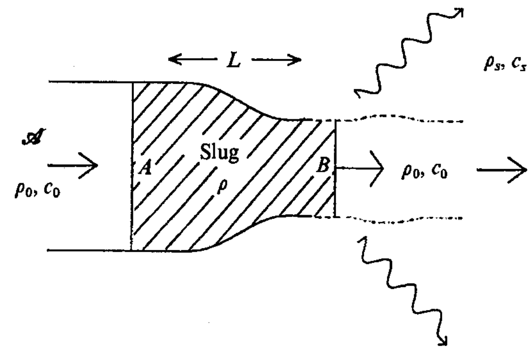

FIGURE 6. Illustrating the idealized model used to determine the acoustic radiation generated during the ejection of a slug from a nozzle.

i.e. approximately

$$\phi \simeq \frac{-a^3}{2|\mathbf{x}|}\frac{\partial}{\partial t}\int (V_j - U_j)\left\{\frac{\partial F}{\partial y_j} + \frac{\mathbf{x}}{|\mathbf{x}|}\cdot\frac{\partial \mathbf{K}}{\partial y_j}\right\}\delta(\mathbf{y} - \mathbf{x}_0(\tau))\,\delta(t - \tau - |\mathbf{x}|/c_s)\,d^3\mathbf{y}\,d\tau. \quad (4.4)$$

Hence the scattered acoustic pressure is given by

$$p \simeq \frac{a^3\rho_s}{2|\mathbf{x}|}\frac{V_k}{c_s}\frac{\partial}{\partial y_k}\left[V_i\frac{\partial}{\partial y_i}\left\{(V_j - U_j)\left(\frac{\partial F}{\partial y_j} + \frac{\mathbf{x}}{|\mathbf{x}|}\cdot\frac{\partial \mathbf{K}}{\partial y_j}\right)\right\}\right] \quad (4.5)$$

evaluated at the retarded position $\mathbf{y} = \mathbf{x}_0(t - |\mathbf{x}|/c_s)$. It is clear from this result that the functions F and $\mathbf{K}$ specify the monopole and dipole characteristics of the radiation, which in terms of the order of magnitude is expressed by

$$p = O\left\{\frac{l}{|\mathbf{x}|}\left(\frac{a}{l}\right)^3 M_s \rho_s U^2\right\}, \quad (4.6)$$

where $M_s \simeq U/c_s$ and l is a length of the order of the nozzle radius.

Next consider the case of a slug exhausting at low Mach number from the nozzle exit (figure 6). We shall assume that the material of the slug lies just within the jet flow in order to avoid the difficulty of taking account of the shear layer between the slug and the ambient atmosphere. In the absence of the slug conditions are steady even in the presence of the vortex layer, which modifies only slightly (?) the radiation properties of the long-wavelength sound.

As in §2 the velocity potential within the nozzle region can be represented by

$$\phi = m(t) + [U_2 + u(t)]\,\phi^*(\mathbf{x}), \quad (4.7)$$

where $m(t)$ is discontinuous across the boundaries of the slug. Surround the slug by a control surface Σ, and in the low Mach number approximation retain only the space-gradient source terms of (2.5).

Thus, in the notation of §2 we have

$$\psi_d \simeq \frac{-1}{4\pi|\mathbf{x}|}\int \frac{\partial}{\partial y_j}\left(\phi\frac{\partial H}{\partial y_j}\right)\delta\left\{t - \tau - \frac{[|\mathbf{x} - \mathbf{K}| - F]}{c_s}\right\}d^3\mathbf{y}\,d\tau$$

$$\simeq \frac{+1}{4\pi|\mathbf{x}|}\frac{1}{c_s}\frac{\partial}{\partial t}\int \phi\frac{\partial H}{\partial y_j}\left\{\frac{\partial F}{\partial y_j} + \frac{\mathbf{x}}{|\mathbf{x}|}\cdot\frac{\partial \mathbf{K}}{\partial y_j}\right\}\delta\left\{t - \tau - \frac{|\mathbf{x}|}{c_s}\right\}d^3\mathbf{y}\,d\tau, \quad (4.8)$$

where $\dot{\phi}$ is to be evaluated just *outside* the boundaries of the slug. Observe that the portion of the surface integral involved in this result which is taken over the solid part of the slug boundary vanishes identically in virtue of (4.2).

Now in a first approximation $\dot{\phi}$ assumes a constant value over the rear face of the slug at station A, and over the exterior surface B of the slug, which has already emerged from the orifice into an effectively constant pressure region. Also, if $\mathbf{l}$ is a unit outward normal to the bounding surface of the slug and dS is a surface element, then

$$\int_A l_j \left[\frac{\partial F}{\partial y_j} + \frac{\mathbf{x}}{|\mathbf{x}|}\cdot\frac{\partial \mathbf{K}}{\partial y_j}\right] dS = -\int_B l_j \left[\frac{\partial F}{\partial y_j} + \frac{\mathbf{x}}{|\mathbf{x}|}\cdot\frac{\partial \mathbf{K}}{\partial y_j}\right] dS, \tag{4.9}$$

since the expression in square brackets is a harmonic function of $\mathbf{y}$. But this also implies that each of these integrals can be replaced by an analogous integral over a cross-section of the duct lying far upstream of the orifice, and where $\partial \mathbf{K}/\partial y_j \to 0$ and $\partial F/\partial \mathbf{y} \to (c_s/c_0, 0, 0)$.

Thus the integral on the left of (4.9) is precisely equal to $-\mathscr{A}c_s/c_0$, and (4.8) becomes

$$\psi_d \simeq \frac{\mathscr{A}}{4\pi|\mathbf{x}|}\frac{1}{c_0}\frac{\partial}{\partial t}[\dot{\phi}(\mathbf{y}, t-|\mathbf{x}|/c_s)]_A^B. \tag{4.10}$$

As before, the leading approximation to this expression is

$$\psi_d \simeq \frac{\mathscr{A}}{4\pi|\mathbf{x}|}\cdot\frac{1}{c_0}\frac{\partial}{\partial t}\left[\frac{\rho-\rho_0}{\rho\rho_0}(p_A - p_B)\right], \tag{4.11}$$

where ρ_0 is the density of the ambient jet flow and ρ is the density of the slug.

Actually this formula is valid before the slug emerges from the orifice, provided that p_B is taken as the local mean flow pressure at the front face of the slug. After emergence p_B may be assumed to equal the constant free-space pressure.

A similar calculation reveals that the monopole contribution ψ_m is $O(M_0)$ smaller than (4.11), so that the leading approximation to the acoustic pressure field radiated into free space is given by

$$p \simeq -\frac{\mathscr{A}}{4\pi|\mathbf{x}|}\left(\frac{\rho_s}{\rho_0}\right)\left(\frac{\rho-\rho_0}{\rho}\right)\frac{1}{c_0}\frac{\partial}{\partial t}\{p_A - p_B\}. \tag{4.12}$$

Thus pressure waves are generated provided that either p_A or p_B is variable. Note that (4.12) is similar in form to the analogous result (3.5), in which the source of the radiation is a contraction in the duct far upstream of the nozzle orifice.

It is clear that, when the length of the slug exceeds the scale L of the nozzle, the sound radiation consists of two equal pulses of opposite sign. The radiation actually commences when the leading face of the slug enters the region of variable mean flow close to the orifice. In this phase p_A is constant and

$$\partial p_B/\partial t \equiv [\mathbf{U}\,.\,\partial p_B/\partial \mathbf{x}]$$

evaluated at the retarded position of the leading face, giving

$$p \sim \frac{\mathscr{A}}{4\pi|\mathbf{x}|}\left(\frac{\rho_s}{\rho_0}\right)\left(\frac{\rho-\rho_0}{\rho}\right)\left[\frac{\mathbf{U}}{c_0}\cdot\nabla p_B\right]. \tag{4.13}$$

In the second phase the time history of the radiation is identical in form with (4.13) but opposite in sign, and occurs when the rear face enters the region of variable mean flow. Thus in both phases the pressure perturbation varies as $(\rho-\rho_0)M_0U^2$, and it will be recognized that we have here precisely the interaction radiation mechanism of (1.4).

In §2 we also considered the possibility of a significant acoustic contribution arising from the differential volumetric expansion of the slug, and concluded that it is important only in the case of a highly compressible slug, exemplified by an air bubble in a hose, and then only for rather large flow velocities. Using the volumetric source term on the right of (2.18) we have in the present case of nozzle flow a free-space pressure perturbation whose order of magnitude is given by

$$p \sim \frac{-\Delta}{4\pi L|\mathbf{x}|}\left(\frac{\rho_s}{\rho}\right)\left[\frac{U^2}{c^2}\frac{\partial p}{\partial x_1}\right], \tag{4.14}$$

where Δ is the volume of the slug. As in the duct-flow case treated in §2, this is comparable with the dipole radiation (4.13) provided that

$$U/c_0 \gtrsim (c/c_0)^2,$$

a condition which, as already noted, is unlikely to be satisfied in the case of air bubbles exhausting from a domestic hose.

Note, however, that the pressure fluctuations given by the dipole term (4.13) decrease to zero as the speed of sound in the jet flow tends to infinity, i.e. when the fluid of the jet is incompressible. This is because an incompressible flow upstream of the slug responds instantaneously to a variation in the thrust without requiring a corresponding variation in the flow velocity, which is necessary if sound is to escape from the orifice of the nozzle into free space. In this limiting case the volumetric source term must dominate the radiation field.

In order to illustrate the magnitudes of the noise levels predicted by the present theory, consider first the case of an air bubble exhausting from a hose 1·5 cm in diameter having a nozzle contraction scale L of 2·5 cm. Assume further that the mean flow speed U is typically of order 10 m s^{-1} and that the pressure drop across the nozzle is 1 atmos. Then (4.13) predicts a sound pressure level of about 80 dB at a distance of 1 m from the nozzle.

As a second example consider the case in which the slug consists of an entropy inhomogeneity in a cold air jet. Assume that $(\rho-\rho_0)/\rho \simeq 0{\cdot}03$, corresponding to a 3 % temperature fluctuation in the duct. Take the mean flow Mach number in the duct to be 0·35, and suppose that the jet pipe has a diameter of 10 cm with a contraction scale $L \simeq 4$ cm where the mean pressure decreases by 2 atmos. Then it follows from (4.13) that at a distance of 10 nozzle diameters from the nozzle the perturbation sound pressure level $\sim$ 128 dB.

5. Conclusion

This paper has examined the sound generated when a density inhomogeneity is convected in a low Mach number steady flow through a contraction in a duct, and also when the inhomogeneity exhausts from a nozzle. Two extreme possibilities have been discussed: the first is that in which the inhomogeneity consists of a slug of fluid of different density from the ambient mean flow, and the second that of a small spherical pellet.

The velocity and Mach number dependence of the sound generated by each of these models is generally that appropriate to a *dipole* source. The pellet behaves as a point dipole during the period in which it is slipping through the ambient fluid in the region of non-uniform mean flow; the slug gives rise to a fluctuation in thrust which is again equivalent to a dipole source. The one exception arises in the case of a highly compressible slug convected at high speed in a relatively incompressible ambient mean flow. The work done on the ambient fluid during the large expansion of the slug as it convects into a region of lower ambient pressure is radiated as sound, and is comparable with that due to the fluctuation in thrust provided that the mean flow Mach number exceeds the square of the ratio of the speed of sound in the slug to that in the ambient flow.

At a contraction the dipole perturbation pressure levels developed within the duct are $O(\rho_0 U^2)$, where ρ_0 is the density of the mean flow. In the case of a slug or a pellet exhausting into free space from a nozzle, the radiation is always dominated at low Mach numbers by that due to the fluctuation in thrust, and is proportional to $\rho_s U^3/c_0$, where ρ_s and c_0 are respectively the density in free space and the speed of sound in the jet flow.

This paper documents a study conducted as part of the Rolls Royce (1971) Ltd research programme on high-speed jet noise.

Appendix. Low frequency approximation to the duct Green's function

To determine the solution of (2.3) appropriate to low frequency source distributions, we first consider the equation

$$c_0^{-2}(\partial/\partial t + \mathbf{U}\,.\,\partial/\partial \mathbf{x})^2\,\Theta - \nabla^2\Theta = e^{-i\omega t}\delta(\mathbf{x}-\mathbf{y}), \tag{A 1}$$

in which it is assumed that the source point $\mathbf{y}$ is located in the region of the duct contraction of figure 1. The analysis is facilitated by consideration of the reciprocal problem

$$c_0^{-2}(\partial/\partial t - \mathbf{U}\,.\,\partial/\partial \mathbf{x})^2\,\overline{\Theta} - \nabla^2\overline{\Theta} = e^{-i\omega t}\delta(\mathbf{x}-\bar{\mathbf{x}}), \tag{A 2}$$

in which the irrotational convection velocity $\mathbf{U}$ is *reversed* at all points of the mean flow, and the source point $\bar{\mathbf{x}}$ is now located far downstream of the contraction. Then by a reverse-flow reciprocal theorem established in Howe (1975),

$$\Theta(\bar{\mathbf{x}}) = \overline{\Theta}(\mathbf{y}). \tag{A 3}$$

Now for sufficiently small values of the radian frequency ω, the disturbance generated by the point source in (A 2) will have developed into a plane propagating incident wave on reaching the duct contraction. The potential of this incident wave in the reciprocal problem is given by

$$\overline{\Theta}_I \equiv \overline{\Theta}_I\left(t+\frac{x_1}{c_0(1+M_2)}\right) = \frac{ic_0}{2\omega A_2}\exp\left\{\frac{i\omega\bar{x}_1}{c_0(1+M_2)} - i\omega\left[t+\frac{x_1}{c_0(1+M_2)}\right]\right\}. \tag{A 4}$$

At the contraction a reflected wave $\overline{\Theta}_R$ and a transmitted wave $\overline{\Theta}_T$ are generated. Far from the contraction these have the following respective functional forms:

$$\left.\begin{aligned}\overline{\Theta}_R &\equiv \overline{\Theta}_R\left(t-\frac{x_1}{c_0(1-M_2)}\right), \quad x_1 > 0,\\ \overline{\Theta}_T &\equiv \overline{\Theta}_T\left(t+\frac{x_1}{c_0(1+M_1)}\right), \quad x_1 < 0.\end{aligned}\right\} \tag{A 5}$$

In the approximation in which terms of order $(\omega L/c_0)^2$ and higher are neglected in the low frequency expansion of the Green's function, the flow in the vicinity of the contraction is incompressible, and the potential there is given by

$$\overline{\Theta} = \chi_0(t) + \chi_1(t)\,\phi^*(\mathbf{x}), \tag{A 6}$$

where the second term on the right is $O(\omega L/c_0)$ relative to the first. The function $\phi^*(\mathbf{x})$ is harmonic and describes steady irrotational flow through the contraction. It is normalized such that

$$\overline{\Theta} \rightarrow \left\{\begin{aligned}&\chi_0(t)+\chi_1(t)\,x_1 \quad \text{as} \quad x_1 \rightarrow +\infty,\\ &\chi_0(t)+\chi_1(t)\left\{\frac{A_2}{A_1}x_1 - \frac{A_1}{K}\right\} \quad \text{as} \quad x_1 \rightarrow -\infty,\end{aligned}\right\} \tag{A 7}$$

where K is the 'conductivity' of the contraction.

These asymptotic expressions must match the corresponding terms in the expansion of the acoustic wave field for small retarded times $x_1/c_0(1 \pm M_{1,2})$, and this implies that

$$\left.\begin{aligned}\chi_0(t) &= \overline{\Theta}_I(t)+\overline{\Theta}_R(t),\\ \chi_0(t)-(A_1/K)\,\chi_1(t) &= \overline{\Theta}_T(t)\end{aligned}\right\} \tag{A 8}$$

and for $M^2 \ll 1$,

$$\left.\begin{aligned}\chi_1(t) &= -\frac{i\omega}{c_0}(1-M_2)\,\overline{\Theta}_I(t)+\frac{i\omega}{c_0}(1+M_2)\,\overline{\Theta}_R(t),\\ \chi_1(t) &= -\frac{i\omega}{c_0}\frac{A_1}{A_2}(1-M_1)\,\overline{\Theta}_T(t).\end{aligned}\right\} \tag{A 9}$$

Solving for χ_0 and χ_1 we ultimately find that in the vicinity of the contraction

$$\begin{aligned}\overline{\Theta}(\mathbf{x},t) &= \frac{2A_2}{A_1+A_2}\left\{1-\frac{i\omega}{c_0}\left[\frac{A_1}{K}\frac{M_2^2}{M_1+M_2}+\frac{A_1}{A_2}\frac{\phi^*(\mathbf{x})}{1+M_1}\right]+\ldots\right\}\overline{\Theta}_I(t)\\ &\simeq \frac{ic_0}{\omega(A_1+A_2)}\exp\left\{-i\omega\left[t-\frac{\bar{x}_1}{c_0(1+M_2)}+\frac{A_1M_2^2}{c_0K(M_1+M_2)}+\frac{A_1\phi^*(\mathbf{x})}{c_0A_2(1+M_1)}\right]\right\},\end{aligned} \tag{A 10}$$

use having been made of (A 4).

By the reverse-flow theorem embodied in (A 3) this is also the potential $\Theta(\bar{\mathbf{x}}, t)$ at $\bar{\mathbf{x}}$ due to a harmonic point source located at $\mathbf{x}$. The corresponding approximation to the Green's function $G(\mathbf{x}, \mathbf{y}; t, \tau)$ generated by a source

$$\delta(\mathbf{x}-\mathbf{y})\,\delta(t-\tau)$$

located at $\mathbf{y}$ in the original problem of (2.3) is obtained by replacing $\mathbf{x}$ by $\mathbf{y}$, and $\bar{\mathbf{x}}$ by $\mathbf{x}$ in (A 10), multiplying by $(2\pi)^{-1}\exp(i\omega\tau)$ and integrating over all ω, the causality condition being satisfied by indenting the path of integration to pass above the singularity at $\omega = 0$. Noting that at the observation point

$$x_1 \gg (A_1/K)\, M_2,$$

we immediately obtain the result (2.4).

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